Indian Statistical Institute
B.Math.(Hons.) I Year
Second Semester Mid Semester Examination, 2004-2005
Algebra II

Time: 3 hrs | Date:11-03-05 | Total Marks : 50

Attempt all questions

1. State whether the following are true or false and justify your answers.
   a) $I\!R[x, y]$ is an Euclidean domain. [3 marks]
   b) $x^2 + y^2 + 1$ is irreducible in $Q[x, y]$. [3 marks]
   c) $\mathbb{Z}[x]/(2x - 6, x - 10)$ is an integral domain. [4 marks]
2. Let k be a field and let $A = k[x, y, z]/(xy - z^2)$ where x, y, z are variables. Let $\bar{x}, \bar{y}, \bar{z}$ denote the images of x, y, z respectively in A. Prove that the ideal $I = (\bar{x}, \bar{z})$ is a prime ideal of A. [10 marks]
3. Let A be a commutative ring with identity and let $\mathcal{M}$ be a maximal ideal of A such that every element of the form $1 + x$ for $x \in \mathcal{M}$ is a unit in A. Show that $\mathcal{M}$ is the unique maximal ideal of A (i.e.$(A, \mathcal{M})$ is a local ring). [10 marks
4. Let m and n be two integers. Prove that their greatest common divisor in $\mathbb{Z}$ is the same as their greatest common divisor $\mathbb{Z}[i]$. [10 marks]
5. Prove that $x^n + x^{n-1} + \ldots + x^2 + x + 1$ is irreducible over $\mathbb{Z}$, if and only if, n is a prime. [10 marks]